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FREE SUBALGEBRAS OF LOOP SPACE HOMOLOGY AND MASSEY PRODUCTS

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LET X be a path-connected space with a base point x_0. Let R be a ring with 1. For every p-chain (or p-cochain) v, set $Jv = (-1)^p v$. The main result in this paper is that, if a cochain $w \in C^+(X; R)$ satisfies the condition $\delta x = Jw \cup w$, then it induces a ring homomorphism

$$H_*(\Omega X) \to R.$$

Observe that w can be taken as a twisting cochain. (See [2] and [6].) If X is simply connected then it follows essentially from a Brown's result [2, §10] that the twisting cochain w gives rise to a ring homomorphism

$$H_*(\Omega X) \approx H_*(F) \to R,$$

where F is the cobar construction on the singular simplicial chain complex of X with a single vertex at x_0. We shall bypass the use of the cobar construction through a direct method and are therefore able to remove the requirement of simple connectedness.

For applications, we limit ourselves mainly to the case of $R = k[[\mathrm{X}]] = k[[X_1, \ldots, X_m]]$ being the algebra of formal power series in noncommutative indeterminates $X_1, \ldots, X_m$ over a commutative ring k with 1. A twisting cochain will take the form

$$w = \sum w_i X_i + \sum w_{ij} X_i X_j + \cdots$$

where $w_i, w_{ij}, \ldots$ are cochains with coefficient in k. The condition $\delta w = Jw \cup w$ means that

$$\delta w_i = 0, \ \delta w_{ij} - Jw_i \cup w_j = 0,$$
$$\delta w_{ijk} - Jw_i \cup w_{jk} - Jw_{ij} \cup w_k = 0 \text{ etc.}$$

Observe that these are precisely defining relations for a family of zero Massey products of the cocycles $w_1, \ldots, w_m$ (see [8]). In particular, vanishing of all possible Massey products of the cohomology classes of $w_1, \ldots, w_m$ will assure the existence of a twisting cochain w, for which $w_1, \ldots, w_m$ are presssigned. The resulting algebra homomorphism $H_*(\Omega X; k) \to k[[\mathrm{X}]]$ gives rise to free subalgebras of $H_*(\Omega X; k)$.

With key lemmas in §2, we construct a ring homomorphism $H_*(\Omega X) \to R$ from a twisting cochain in §3. We discuss, in §4, the existence of twisting cochains in the case of

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R being the completion of a connected algebra and specialize, in §5, to the particular case of $R = k[[X]]$. Applications are given in §6–7.

Perhaps it should be pointed out that our work is motivated by a differentiable version as presented in [4]. It is a pleasure to thank A. Tsuchiya for valuable suggestions.

§1.

Denote by $Q(X)$ the normalized singular cubic chain complex of X. Recall that a (singular) cube $I^n \to X$, $n > 0$, is degenerate, if the map does not depend on one of the coordinates $(\xi_1, \ldots, \xi_n)$ of I^n. Let x_0 also denote the 0-cube at x_0.

Let $u: I^n \to X$ be a singular cube. If $H = (h_1, \ldots, h_r)$ is an ordered subset of the ordered set $(1, \ldots, n)$, let the singular r-cube $\lambda_H{}^\varepsilon u: I^r \to X$, $\varepsilon = 0, 1$, be given by

$$(\lambda_H{}^\varepsilon u)(t_{h_1}, \ldots, t_{h_r}) = u(\tau_1, \ldots, \tau_n)$$

where $\tau_i = \varepsilon$ for $i \notin H$ and $\tau_i = t_i$ for $i \in H$. For the diagonal map, we use

$$\Delta\colon Q(X) \to Q(X) \otimes Q(X)$$

given by $u \mapsto \sum\limits_H \varepsilon_{HK} \lambda_H{}^0 u \otimes \lambda_K{}^1 u$ summing over all ordered subsets H of $(1, \ldots, n)$, where K is the ordered complement of H, and ε_{HK} is the sign of the permutation (HK) (see [7]).

Let $S\colon Q(\Omega X) \to Q(X)$ be the loop suspension such that, for each p-cube $\alpha\colon I^p \to \Omega X$, the $(p+1)$-cube

$$S\alpha\colon I \times I^p \to X$$

is given by $(t, \xi) \mapsto \alpha(\xi)(t)$. Then $S\partial = -\partial S$. We verify the next assertion.

LEMMA 1.1. *The composite homomorphism ΔS is given by*

$$\alpha \mapsto x_0 \otimes S\alpha + S\alpha \otimes x_0.$$

Write $C^*(X; G) = C(Q(X); G)$. Set $C^+(X; G) = \sum\limits_{r>0} C^r(X; G)$.

LEMMA 1.2. *Let R be a ring. If $w \in C^+(X; R)$ is such that*

$$\delta\Delta \in C^+(X; R) \cup C^+(X; R),$$

then S^*w is a cocycle in $C^*(\Omega X; R)$.

Proof. $\langle \delta S^* w, \alpha\rangle = -\langle \delta w, S\alpha\rangle = -\langle \sum w_i' \otimes w_i'', \Delta S\alpha\rangle$ which vanishes owing to the preceding lemma and the fact that $w_i', w_i'' \in C^+(X; R)$.

§2.

Let $U\colon Q(\Omega X) \otimes Q(\Omega X) \to Q(X)$ be the graded homomorphism of degree 2 such that if α and β are p- and q-cubes of ΩX, then

$$u = U(\alpha \otimes \beta)\colon I \times I \times I^p \times I^q \to X$$

is the $(p+q+2)$-cube given by

$$u(s, t, \xi, \eta) = \begin{cases} \alpha(\xi)(t/(1 - \frac{1}{2}s)), & \text{for} \quad 0 \leqq t \leqq 1 - \frac{1}{2}s, \\ \beta(\eta)(2t + s - 2), & \text{for} \quad 1 - \frac{1}{2}s \leqq t \leqq 1. \end{cases}$$

LEMMA 2.1. *If* $u = U(\alpha \otimes \beta)$, *then*

$$\Delta u = x_0 \otimes u + (JS\alpha) \otimes S\beta + u \otimes x_0 .$$

Proof. Let H be an ordered subset of $(1, \ldots, p+q+2)$, and K, the ordered complement of H. Set $v_{HK} = \lambda_H{}^0 u_K \otimes \lambda_K{}^1 u$. If H(resp. K) is empty, then

$$v_{HK} = x_0 \otimes u \text{ (resp. } u \otimes x_0).$$

Assuming that neither H nor K is empty, verify that

(*a*) if $2 \in K$, then $v_{HK} = 0$,
(*b*) if $1 \in H$, $2 \in H$, then $v_{HK} = 0$,
(*c*) if $1 \in K$ and $2 \in H$, then $v_{HK} = 0$ unless

$$H = (2, \ldots, p, p+2) \text{ and } K = (1, p+3, \ldots, p+q+2),$$

for which case $v_{HK} = S\alpha \otimes S\beta$ and $\varepsilon_{HK} = -(-1)^p$.

Let $\alpha \cdot \beta \colon I^p \times I^q \to \Omega X$ be the $(p+q)$-cube such that $(\xi, \eta) \to \alpha(\xi)\beta(\eta)$, which denotes the product of the loops $\alpha(\xi)$ and $\beta(\eta)$.

LEMMA 2.2. *If* $u = U(\alpha \otimes \beta)$, *then*

$$\partial u = U(\partial\alpha \otimes \beta + J\alpha \otimes \partial\beta) - \langle 1, \beta\rangle S\alpha + S(\alpha \cdot \beta) - \langle 1, \alpha\rangle S\beta.$$

Proof. The face $\lambda_1{}^0 u$ is always degenerate unless $q = 0$, for which case $\lambda_1{}^0 u = S\alpha$. For any q, $\lambda_1{}^1 u = S(\alpha \cdot \beta)$. The face $\lambda_2{}^0 u$ is always degenerate. So is the face $\lambda_2{}^1 u$, unless $p = 0$, for which case it is equal to $S\beta$.

For $2 < i \leqq p+2$, $\lambda_i{}^\varepsilon u = U(\lambda_{i-2}^\varepsilon \alpha \otimes \beta)$ and, for $p+2 < i \leqq p+q+2$,

$$\lambda_i{}^\varepsilon u = U(\alpha \otimes \lambda_{i-p-2}^\varepsilon \beta).$$

Note also that $\langle 1, \alpha\rangle = \delta_{0p}$ and $\langle 1, \beta\rangle = \delta_{0q}$.

COROLLARY. *If* z_1 *and* z_2 *are cycles of* $Q(\Omega X)$, *then*

$$\partial U(z_1 \otimes z_2) = S(-\langle 1, z_2\rangle z_1 + z_1 \cdot z_2 - \langle 1, z_1\rangle z_2).$$

§3.

THEOREM. *Let R be a ring with* 1. *If* $w \in C^+(X; R)$ *is such that* $\delta w - Jw \cup w = 0$, *then there is a ring homomorphism*

$$T\colon H_*(\Omega X) \to R$$

given by $[z] \to \langle 1 + S^*w, z\rangle$ *for every cycle* z *of* $Q(\Omega X)$.

Proof. Let $u = U(z_1 \otimes z_2)$. According to Lemma 2.1,

$$\langle w, \partial u\rangle = \langle Jw \cup w, u\rangle = \langle Jw \otimes w, JSz_1 \otimes Sz_2\rangle = \langle S^*w, z_1\rangle\langle S^*w, z_2\rangle.$$

On the other hand, according to Corollary, Lemma 2.2,

$$\langle w, \partial u\rangle = -\langle S^*w, z_1\rangle\langle 1, z_2\rangle + \langle S^*w, z_1 \cdot z_2\rangle - \langle 1, z_1\rangle\langle S^*w, z_2\rangle.$$

Hence

$$\langle 1 + S^*w, z_1 \cdot z_2 \rangle = \langle 1 + S^*w, z_1 \rangle \langle 1 + S^*w, z_2 \rangle.$$

Remark. The above theorem can be further generalized. Assume that R is equipped with an additive endomorphism d: $R \to R$ such that $d^2 = 0$. Assume also that $H_*(R) = \operatorname{Ker} d/\operatorname{Im} d$ is also a ring whose multiplication is induced by that of R. We call $w \in C^+(X; R)$ a twisting cochain if $\delta w - dw = Jw \cup w$. The theorem then reads as follows:

Every twisting cochain $w \in C^+(X; R)$ induces a ring homomorphism

$$T: H_*(\Omega X) \to H_*(R)$$

such that $[z] \mapsto$ the homology class of $\langle 1 + S^*w, z \rangle$.

§4.

Let k be a commutative ring with 1. Let $\{A_r\}_{r \geq 0}$ be a connected graded k-algebra. We are now interested in the case where R is the ring of all formal infinite sums

$$a = a_0 + \cdots + a_r + \cdots, \qquad a_r \in A_r.$$

Let R^+ consist of all those a with $a_0 = 0$. Then every element $C^+(X; R^+)$ can be written as formal infinite sum

$$w = w_1 + \cdots + w_r + \cdots, \qquad w_r \in C^+(X; A_r)$$

so that $\delta w - Jw \cup w$ is equal to

$$\delta w_1 + (\delta w_2 - Jw_1 \cup w_1) + \cdots + \Big(\delta w_r - \sum_{1 \leq i < r} Jw_i \cup w_{r-i}\Big) + \cdots.$$

If each w_i is of degree $\equiv 1 \bmod q$, then each term of $\delta w - Jw \cup w$ is of degree $\equiv 2 \bmod q$.

If $\delta w - Jw \cup w = 0$, then w_1 must be a cocycle. Moreover, Theorem 3 provides a ring homomorphism

$$T: H_*(\Omega X) \to R \tag{4.1}$$

given by $z \mapsto \langle 1, z \rangle + \langle w_1, Sz \rangle + \cdots + \langle w_r, Sz \rangle + \cdots$.

The next lemma follows from a simple dimension argument.

LEMMA. *Let $w_1 \in \sum_{s \geq 0} C^{sq+1}(X; A_1) \subset C^+(X; A_1)$. If $H^{sq+2}(X; A_r) = 0$ for $s \geq 0$ and $r \geq 2$, then there exist w_r, $r \geq 2$, such that $w = w_1 + \cdots + w_r + \cdots$ satisfies $\delta w - Jw \cup w = 0$.*

§5.

Let $X_1, \ldots, X_m$ be noncommutative indeterminates. Let A_r denote the free k-module on the monomials $X_{i_1} \cdots X_{i_r}$ of degree r. Then $R = k[[X]]$ is the algebra of formal power series of the noncommutative indeterminates $X_1, \ldots, X_m$ over k.

We say that a monomial u' in $k[[X]]$ precedes another monomial $u = X_{i_1} \cdots X_{i_r}$ if u' is a proper factor of u, i.e. u' is of the type $X_{i_\lambda} X_{i_{\lambda+1}} \cdots X_{i_\mu}$ of length $<r$. A leading term of a formal power series in $X_1, \ldots, X_m$ is a nonvanishing term such that all preceding terms vanish.

For any $w \in C^+(X; k[[\mathrm{X}]]^+)$, the coefficient of the monomial $u = X_{i_1} \cdots X_{i_r}$ in $\varkappa = \delta w - Jw \cup w$ is

$$\varkappa_{i_1 \cdots i_r} = \delta w_{i_1 \cdots i_r} - \sum Jw_{i_1 \cdots i_\lambda} \cup w_{i_{\lambda+1} \cdots i_r}.$$

If u represents a leading term of $\varkappa$, then the vanishing terms that precede the term of u give rise to a defining system for the Massey product $\langle w_{i_1}, \ldots, w_{i_r} \rangle$. All defining systems for Massey products can be obtained in this way.

In the case of $\varkappa_{i_1 \cdots i_r} = 0$, Lemma 1.2 implies that $S^* w_{i_1 \cdots i_r}$ represents a cohomology class in $H^*(\Omega X; k)$. D. Kraines has constructed such cohomology classes in a somewhat different setting [9].

LEMMA. *Let $w_1, \ldots, w_m$ be cocycles of $C^+(X; k)$. If all possible Massey products of $w_1, \ldots, w_m$ contain zero only, then there exists $w \in C^+(X; k[[\mathrm{X}]]^+)$ of the type*

$$w = \sum w_i X_i + \cdots$$

such that $\delta w - Jw \cup w = 0$.

Observe that if $w_1, \ldots, w_m$ are all of degree $\equiv 1 \bmod q$ for some positive integer q, then the Massey products of $w_1, \ldots, w_m$ lie in $\sum_{k \geqq 0} H^{rq+2}(X; k)$.

§6.

THEOREM. *Let k be a commutative ring with 1. Let $w = \sum w_i X_i + \sum w_{ij} X_i X_j + \cdots$ be a cochain in $C^+(X; k[[\mathrm{X}]])$ such that $\delta w - Jw \cup w = 0$. Let $z_1, \ldots, z_m \in H_*(\Omega X; k)$. If $\langle w_i, Sz_j \rangle = \delta_{ij}$, $i, j = 1, \ldots, m$, then there exists a k-algebra homomorphism*

$$\Theta \colon H_*(\Omega X; k) \to k[[\mathrm{X}]]$$

such that $z_i \mapsto \langle 1, z_i \rangle + X_i$.

Proof. Let T be given by (4.1). Then

$$Tz_i = \langle 1, z_i \rangle + X_i + \text{higher degree terms}.$$

Let χ be the obvious endomorphism of $k[[\mathrm{X}]]$ with

$$\chi X_i = Rz_i - \langle 1, z_i \rangle.$$

Then χ^{-1} can be constructed by comparing the coefficients occurring in the identities $\chi^{-1}\chi X_i = X_i$, $i = 1, \ldots, m$. Hence $\Theta = \chi^{-1} T$ is the desired homomorphism.

COROLLARY 1. *Let q be a given positive integer. Let X be a space such that $H^{rq+2}(X; k) = 0$, $r \geqq 0$. If $z_i \in H_*(\Omega X; k)$, $i = 1, \ldots, m$, are of degree $\equiv 0 \bmod q$ and $w_j \in H^+(X; k)$, $j = 1, \ldots, m$, are of degree $\equiv 1 \bmod q$ such that $\langle w_i, Sz_i \rangle = \delta_{ij}$, then $z_1, \ldots, z_m$ freely generate a free subalgebra of the k-algebra $H_*(\Omega X; k)$.*

For example, let $X = S_1 \vee \cdots \vee S_m$ where each S_i is a copy of S^{q+1}. If w_i is the generator of $H^{q+1}(S_i)$ and if z_i is the generator of $H_q(\Omega S_i)$, then the above corollary implies that $H_*(\Omega X)$ contains a free subring generated by $z_1, \ldots, z_l$. This is a known fact, which is a special case of a theorem due to Bott and Samelson [1].

For the case of $q = 0$, let us assume that $z_1, \ldots, z_l$ belong to the image of

$$\pi_1(X) = \pi_0(\Omega X) \to H_0(\Omega X) \to H_0(\Omega X; k)$$

so that $z_1, \ldots, z_l$ are units of the k-algebra $H_*(\Omega X; k)$, and

$$\Theta z_i^{-1} = 1 - X_i + X_i^2 - \cdots.$$

It is due to Magnus that, if k is a field of characteristic 0, then $1 + X_1, \ldots, 1 + \mathrm{X}_l$ form a basis for a free subgroup G of the multiplicative monoids of $k[[\mathrm{X}]]$ (see Theorem 5.6 [10]). With a slight modification of its proof, the theorem is also valid for cases of characteristic $\neq 0$.

COROLLARY 2 (Stallings [12]). *Let k be a field. If $H^2(X; k) = 0$ and if the images of $\alpha_1, \ldots, \alpha_l \in \pi_1(X)$ in $H_1(X; k)$ are linearly independent, then $\alpha_1, \ldots, \alpha_l$ form a basis for a free subgroup of $\pi_1(X)$.*

Remark. Instead of $H^2(X; k) = 0$, the original result of Stallings requires that $H_2(\pi_1(X); k) = 0$. Using a standard procedure of killing higher homotopies or a therorem in [12], one can weaken the condition to $H_2(X; k) = 0$.

For the case of k being a field of characteristic 0, Corollary 1 can be strengthened to a result which we mention as follows:

COROLLARY 3. *Let k be a field of characteristic 0, and q a given positive integer. Let X be a path connected space such that $H^{rq+2}(X; k) = 0$, $r \geqq 0$. Let $z_1, \ldots, z_l \in H_0(\Omega X; k)$ be units of $H_*(\Omega X; k)$, and let $z_{l+1}, \ldots, z_m \in H_+(\Omega X; k)$ be of degree $\equiv 0 \bmod q$. If there exist $w_1, \ldots, w_m \in H^+(X; k)$ such that $\langle w_i, Sz_j \rangle = \delta_{ij}$, then*

(*a*) *$z_1, \ldots, z_l$ form a basis of a free group G of the multiplicative monoid of $H_*(\Omega X; k)$;*
(*b*) *$z_{l+1}, \ldots, z_m$ form a basis for a free algebra A of $H_*(\Omega X; k)$;*
(*c*) *the free product $kG * A$ of the group algebra kG and the algebra A is embedded in $H_*(\Omega X; k)$.*

§7.

Instead of the formal power series algebra $k[[\mathrm{X}]]$, we may use one of its quotient algebras for the coefficient of cohomology. Let us consider a particular case.

Let each indeterminate X_i be given a weight $p_i \geqq 0$. The weight of a monomial $X_{i_1} \cdots \mathrm{X}_{i_r}$ is defined to be $p_{i_1} + \cdots + p_{i_r}$. Let A denote the k-algebra obtained from the k-algebra $k[[\mathrm{X}]]$ by truncating all terms of weight $>$ *a* given natural integer s. The next assertion is obtained in the same manner as Corollary 1, Theorem 6.

PROPOSITION. *Let $q > 0$ and $s \geqq 0$ be given integers. Let X be a space such that $H^{rq+2}(X; k) = 0$ for $0 \leqq r = s/q$. Let $z_i \in H_{p_i}(\Omega X; k)$, $i = 1, \ldots, m$ be such that $q | p_i$ and $p_i \leqq s$. If $w_j \in H^{p_j+1}(\Omega X; k)$, $j = 1, \ldots, m$, are such that $\langle w_i, Sz_j \rangle = \delta_{ij}$, then there is a surjective k-algebra homomorphism*

$$H_*(\Omega X; k) \to A$$

so that $z_i \mapsto \langle 1, z_i \rangle + X_i$, where A is the mentioned truncated formal power series algebra.

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